

# Universal centers in the cubic trigonometric Abel equation

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**Abstract.** We study the center problem for the trigonometric Abel equation  $d\rho/d\theta = a_1(\theta)\rho^2 + a_2(\theta)\rho^3$ , where  $a_1(\theta)$  and  $a_2(\theta)$  are cubic trigonometric polynomials in  $\theta$ . This problem is closely connected with the classical Poincaré center problem for planar polynomial vector fields. A particular class of centers, the so-called universal centers or composition centers, is taken into account. An example of non-universal center and a characterization of all the universal centers for such equation are provided.

**Keywords:** center problem, Abel differential equation, universal center, composition condition, polynomial differential equations.

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## 1 Introduction and statement of the main results

In this note we consider the Abel trigonometric differential equation

$$\frac{d\rho}{d\theta} = a_1(\theta)\rho^2 + a_2(\theta)\rho^3, \quad (1.1)$$

defined on the cylinder  $(\rho, \theta) \in \mathbb{R} \times \mathcal{S}^1$  and where  $a_1(\theta)$  and  $a_2(\theta)$  are real trigonometric polynomials in  $\theta$  of degree  $\max\{\deg a_1, \deg a_2\} = d$ .


Equation (1.1) is a particular case of the analytic ordinary differential equation

$$\frac{d\rho}{d\theta} = \mathcal{F}(\rho, \theta) = \sum_{i \geq 1} a_i(\theta)\rho^{i+1}, \quad (1.2)$$

defined on the cylinder  $(\rho, \theta) \in \mathbb{R} \times \mathcal{S}^1$  and where  $a_i(\theta)$  are real trigonometric polynomials in  $\theta$ . We denote by  $\rho = \rho(\theta; \rho_0)$  the general solution of (1.2) with initial condition  $\rho(0; \rho_0) = \rho_0$ . We remark that  $\rho = 0$  is a particular solution and that, as a consequence, we have that  $\rho(\theta; \rho_0)$  is defined for all  $\theta \in \mathcal{S}^1$  for  $|\rho_0|$  small enough.

We say that equation (1.2) has a *center* when  $\rho(2\pi; \rho_0) = \rho_0$  for  $|\rho_0|$  small enough, that is, when all the orbits in a neighborhood of the particular solution  $\rho = 0$  are  $2\pi$ -periodic. The

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center problem for equation (1.2) is to find conditions on the coefficients  $a_i(\theta)$  under which this equation determines a center. The original center problem arises from the study of the planar analytic differential systems, see for instance [15] and references therein.

Classically, there exist two ways to characterize centers in equation (1.2). The first one is to prove the existence of a first integral  $H(\rho, \theta)$  which is  $2\pi$ -periodic in  $\theta$ . A function  $H(\rho, \theta)$  defined in a neighborhood of  $\rho = 0$ , of class  $C^1$  and non locally constant, is a *first integral* of equation (1.2) if  $H(\rho(\theta; \rho_0), \theta)$  does not depend on  $\theta$ . Equivalently,  $(\partial H / \partial \rho) \mathcal{F}(\rho, \theta) + \partial H / \partial \theta \equiv 0$ .

The second way is to consider the first return map  $P(a)$  associated to equation (1.2)  $P(a)(\rho_0) := \rho(2\pi; \rho_0)$  and to verify that it is the identity map for  $|\rho_0|$  small enough. In [6] (see also [7]), an explicit expression for the first return map  $P(a)(\rho_0)$  was given. We remark that  $P(a)(\rho_0)$  is an absolute convergent power series for sufficiently small initial values  $|\rho_0|$  whose development takes the form

$$P(a)(\rho_0) = \rho_0 + \sum_{n \geq 1} c_n(a) \rho_0^{n+1}. \quad (1.3)$$

**Theorem 1.1.** [7] *For sufficiently small initial values  $|\rho_0|$  the first return map  $P(a)$  is an absolute convergent power series (1.3), where*

$$c_n(a) = \sum_{i_1 + \dots + i_k = n} c_{i_1, \dots, i_k} I_{i_1, \dots, i_k}(a), \quad \text{and}$$

$$c_{i_1, \dots, i_k} = (n - i_1 + 1) \cdot (n - i_1 - i_2 + 1) \cdot (n - i_1 - i_2 - i_3 + 1) \cdots 1,$$

and where  $I_{i_1, \dots, i_k}(a)$  is the following iterated integral of order  $k$

$$I_{i_1, \dots, i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \dots \leq s_k \leq 2\pi} a_{i_k}(s_k) \cdots a_{i_1}(s_1) ds_k \cdots ds_1.$$

Of course, equation (1.2) has a center if and only if  $c_n(a) = 0$ , for all  $n \geq 1$ . From the form of the first return map  $P(a)$ , the following definition, given in [7], follows in a natural way.

**Definition 1.2.** [7] The differential equation (1.2) has a *universal center* if for all positive integers  $i_1, \dots, i_k$  with  $k \geq 1$  the iterated integral  $I_{i_1, \dots, i_k}(a) = 0$ .

The expression of the coefficients of the first return map  $P(a)(\rho_0) := \rho(2\pi; \rho_0)$  for the Abel differential equation  $d\rho/d\theta = a_0(\theta)\rho + a_1(\theta)\rho^2 + a_2(\theta)\rho^3$ , and thus for equation (1.1), was given by [2, 10, 11].

We say that differential equation (1.2) satisfies the *composition conditions* if there is a nonconstant trigonometric polynomial  $q$  and there are polynomials  $p_i \in \mathbb{R}[z]$ , for  $i \geq 1$  such that

$$\tilde{a}_i = p_i \circ q, \quad i \geq 1, \quad \text{where} \quad \tilde{a}_i(\theta) = \int_0^\theta a_i(s) ds.$$

The first time that this definition appears was in the work Alwash and Lloyd [4]. The composition conditions have been studied by several authors in different contexts, see for instance [3, 4, 15] and references therein.

Universal centers of equation (1.2) were characterized in [14] through the following result.

**Theorem 1.3.** [14] *Any center of the differential equation (1.2) is universal if and only if equation (1.2) satisfies the composition conditions.*

In [7] the same result was proved when equation (1.2) has a finite number of terms.

The aim of this work is to study universal and non-universal centers of an Abel differential equation (1.1) in relation with the degree of the trigonometric polynomials  $a_1(\theta)$  and  $a_2(\theta)$ . Recall that equation (1.1) has a universal center when all the iterated integrals  $I_{i_1 \dots i_k}(a) = 0$ , for all  $i_1, \dots, i_k$ . Now, each of the indexes  $i_1, \dots, i_k$  can only take the values 1 or 2. Besides the characterization of universal centers as composition centers for the Abel trigonometric equation (1.1) proved in [7, 14], in [9] another characterization is provided in terms of the vanishing of a finite set of *double moments*. We assume that the minimal common period of  $a_1$  and  $a_2$  is  $2\pi/k$ , with  $k \in \mathbb{N}^+$ .

**Theorem 1.4.** [9] Equation (1.1) has a universal center if and only if for all  $i, j \in \mathbb{N}$  satisfying  $i + j \leq 4d/k - 3$ ,

$$\int_0^{2\pi} \tilde{a}_1^i(s) \tilde{a}_2^j(s) a_2(s) ds = \int_0^{2\pi} a_1(s) ds = 0.$$

These type of integrals are known as the double moments.

It is well-known that not all the centers of equation (1.1), and thus of equation (1.2), are universal, see [1]. Any quadratic system in the plane can be transformed to an Abel equation of the form (1.1) where  $a_1(\theta)$  and  $a_2(\theta)$  are trigonometric polynomials of degree 3 and 6 respectively. Moreover in [14] it is proved that there are centers of a quadratic system which are not universal (for instance the Darboux component except its intersection with the symmetric one). Indeed, in [14] it is proved that these non-universal centers of some quadratic systems give non-universal centers of their associated Abel equation. A previous and different example of a center of an Abel equation which is not universal and where  $a_1(\theta)$  and  $a_2(\theta)$  are also trigonometric polynomials of degree 3 and 6 respectively, is provided in [8]. Hence, the following open problem is established in [15].

**Open problem:** To determine the lowest degree of the trigonometric polynomials  $a_1(\theta)$  and  $a_2(\theta)$  such that the Abel equation (1.1) has a center which is not universal.

In this paper we solve this open problem, see Theorem 1.6. Blinov in [5] proved the following result which shows that the lowest possible degree such that an Abel equation can have a non-universal center is at least 3.

**Proposition 1.5.** [5] All the centers of equation (1.1) when  $a_1(\theta)$  and  $a_2(\theta)$  are trigonometric polynomials of degree 1 and 2 are universal centers and, in consequence, verify the composition condition.

The proof given in [5] (see also [15]) consists in solving the center problem for equation (1.2) with  $a_1(\theta)$  and  $a_2(\theta)$  of degree at most 2 and to check that all the center cases are universal. However, this procedure is unapproachable for higher degrees due to the cumbersome computations needed to solve the center problem. Indeed, Blinov's result solves the center and the universal center problem for Abel differential equations (1.1) up to degree 2. The next equations to be studied are the cubic ones, i.e.  $d = 3$ .

The following result concludes that the lowest degree of a trigonometric Abel equation (1.1) with a non-universal center is 3.

**Theorem 1.6.** The cubic ( $d = 3$ ) trigonometric Abel differential equation

$$\frac{d\rho}{d\theta} = (\cos \theta + 2 \cos 2\theta) \rho^2 + (\sin \theta - \sin 2\theta + \sin 3\theta) \rho^3, \quad (1.4)$$

has a center which is not universal.

The proof of this result is given in Section 2.

There are two big families of universal centers of equation (1.2): when the equation is either  $\alpha$ -symmetric or of separable variables, see definitions below. Given an angle  $\alpha \in [0, \pi)$ , we say that the differential equation (1.2) is  $\alpha$ -symmetric if its flow is symmetric with respect to the straight line  $\theta = \alpha$ . Obviously, this is equivalent to that equation (1.2) is invariant under the change of variables  $\theta \mapsto 2\alpha - \theta$ . Any differential equation (1.2) which is  $\alpha$ -symmetric has a center, due to the symmetry.

A differential equation (1.2) is of *separable variables* if the function on the right-hand side of equation (1.2) splits as product of two functions of one variable, one depending on  $\rho$  and the other on  $\theta$ , that is,  $d\rho/d\theta = a(\theta)b(\rho)$ . In such a case there is only one center condition which is  $\int_0^{2\pi} a(\theta) d\theta = 0$ .

The following result for equation (1.2) is proved in [14].

**Theorem 1.7.** [14] *If the differential equation (1.2) has a center which is either  $\alpha$ -symmetric, or of separable variables, then it is universal.*

For the case of the Abel trigonometric equation (1.1), we give the following result about the universal centers which belong to the classes of  $\alpha$ -symmetric or of separable variables differential equations. To simplify notation, we consider 1 as a prime number.

**Proposition 1.8.** *If the degrees of  $a_1(\theta)$  and  $a_2(\theta)$  are both prime numbers or they are coprime and the Abel differential equation (1.1) has a universal center then the differential equation is either  $\alpha$ -symmetric or of separable variables.*

As a direct consequence of this result, we have that any universal center of equation (1.1) with  $d = 3$  is either  $\alpha$ -symmetric or of separable variables.

This note is organized as follows. Section 2 contains the proofs of the two main results, namely Theorem 1.6 and Proposition 1.8, together with some preliminary results.

## 2 Preliminary results and proofs of the main results

As we have stated in the previous section, a way to characterize that equation (1.2) has a center is to prove the existence of a first integral  $H(\rho, \theta)$  which is defined in a neighborhood of  $\rho = 0$  and it is  $2\pi$ -periodic in  $\theta$ . A function which is closely related to a first integrals is the inverse integrating factor. A function  $V(\rho, \theta)$  defined in a neighborhood of  $\rho = 0$ , of class  $\mathcal{C}^1$  and non locally null, is an *inverse integrating factor* of equation (1.2) if

$$\frac{\partial V}{\partial \rho} \mathcal{F}(\rho, \theta) + \frac{\partial V}{\partial \theta} = \frac{\partial \mathcal{F}}{\partial \rho} V(\rho, \theta)$$

and  $V(\rho, \theta)$  is  $2\pi$ -periodic in  $\theta$ . Given an inverse integrating factor  $V(\rho, \theta)$  of (1.2), one can construct a first integral  $H(\rho, \theta)$  of (1.2) through the following line integral:

$$H(\rho, \theta) = \int_{(\rho_0, \theta_0)}^{(\rho, \theta)} \frac{d\rho - \mathcal{F}(\rho, \theta)d\theta}{V(\rho, \theta)}$$

along any curve connecting an arbitrarily chosen point  $(\rho_0, \theta_0)$  (such that  $V(\rho_0, \theta_0) \neq 0$ ) and the point  $(\rho, \theta)$ . The following result reads for Corollary 5 in [12] written with our notation and our assumptions, see also [13].

**Lemma 2.1.** [12] Let  $V(\rho, \theta)$  be an inverse integrating factor of equation (1.2) whose leading term in the development around  $\rho = 0$ :

$$V(\rho, \theta) = \rho^\mu v(\theta) + o(\rho^\mu),$$

where  $v(\theta) \neq 0$ , is such that either  $\mu = 0$  or  $\mu > 1$  and  $\mu$  is not a natural number, then equation (1.2) has a center, that is  $\rho = 0$  belongs to a continuum of periodic orbits.

Now we are in conditions to prove our first result.

*Proof of Theorem 1.6.* For the particular Abel differential equation (1.4), we denote by  $a_1(\theta) := \cos \theta + 2 \cos 2\theta$ ,  $a_2(\theta) := \sin \theta - \sin 2\theta + \sin 3\theta$  and

$$\tilde{a}_1(\theta) := \int_0^\theta a_1(s) ds, \quad \tilde{a}_2(\theta) := \int_0^\theta a_2(s) ds.$$

We have that the iterated integral

$$\begin{aligned} I_{221}(a) &= \int_{0 \leq s_1 \leq s_2 \leq s_3 \leq 2\pi} a_2(s_3) a_2(s_2) a_1(s_1) ds_3 ds_2 ds_1 \\ &= - \int_0^{2\pi} \tilde{a}_1(s) \tilde{a}_2(s) a_2(s) ds = \frac{\pi}{2}. \end{aligned}$$

Therefore and on account of Theorem 1.3, if equation (1.4) has a center, it cannot be universal. Moreover, the function

$$H(\rho, \theta) := \frac{g^2 - (\cos \theta + \sin \theta - 1)g + 1 - \cos \theta}{g^2 + (\cos \theta + \sin \theta - 1)g + 1 - \cos \theta} \cdot e^{-4g + 2 \arctan\left(\frac{(\cos \theta - \sin \theta - 1)g}{g^2 + \cos \theta - 1}\right)},$$

with

$$g(\rho, \theta) = \sqrt{\frac{1}{\rho} - \sin \theta + \sin 2\theta}$$

is a first integral of equation (1.4). This is because the function  $H(\rho, \theta)$ , for  $\rho > 0$  small enough, is of class  $C^1$ ; is not constant; it is periodic in  $\theta$  of period  $2\pi$ ; and satisfies  $(\partial H / \partial \rho) \mathcal{F}(\rho, \theta) + \partial H / \partial \theta \equiv 0$ . Therefore, equation (1.4) has a center.

Another way to prove this statement is to note that the algebraic function

$$V(\rho, \theta) = \frac{\rho^{3/2} [2 + 2 \sin(2\theta)\rho + (2 - 3 \cos \theta + 2 \cos(2\theta) - \cos(3\theta)) \rho^2]}{2\sqrt{1 - (\sin \theta - \sin(2\theta))\rho}},$$

is an inverse integrating factor of equation (1.4). On account of Lemma 2.1 and since the leading term of the development of  $V(\rho, \theta)$  around  $\rho = 0$  is  $V(\rho, \theta) = \rho^{3/2} + o(\rho^{3/2})$  (that is  $\mu = 3/2$ ) we have that equation (1.4) has a center.  $\square$

Our second result, Proposition 1.8, relies on the degrees of trigonometric polynomials. The following result is Lemma 16 of [14] deals with the relation between degrees of trigonometric polynomials.

**Lemma 2.2.** [14] Let  $A(\theta)$  and  $B(\theta)$  be two trigonometric polynomials of degrees  $d$  and  $\bar{d}$ , respectively. The following statements hold.

- (a) The trigonometric polynomial  $A'(\theta)$  is of degree  $d$ .
- (b) The trigonometric polynomial  $A(\theta)B(\theta)$  is of degree  $d + \bar{d}$ .

- (c) Let  $N(z)$  be a polynomial in  $\mathbb{R}[z]$  of degree  $k$ , then  $N(A(\theta))$  is a trigonometric polynomial of degree  $k$ .

*Proof of Proposition 1.8.* If the Abel differential equation (1.1) has a universal center then we have that  $\tilde{a}_1(\theta)$  and  $\tilde{a}_2(\theta)$  satisfy the composition conditions i.e., there exist a nonconstant trigonometric polynomial  $q(\theta)$  and two real polynomials  $p_1, p_2 \in \mathbb{R}[z]$  such that  $\tilde{a}_1(\theta) = p_1(q(\theta))$  and  $\tilde{a}_2(\theta) = p_2(q(\theta))$ . Let  $d_i = \deg a_i$  for  $i = 1, 2$ . By Lemma 2.2(a), we have that  $d_i = \deg \tilde{a}_i$  for  $i = 1, 2$ .

Assume first that  $d_1$  and  $d_2$  are both prime numbers. Then, by Lemma 2.2(c) we have that either  $\deg q = 1$  or  $\deg p_1 = \deg p_2 = 1$ . In the case that  $\deg q = 1$  the differential equation (1.1) has a center which is  $\alpha$ -symmetric, see [14]. In the case that  $\deg p_1 = \deg p_2 = 1$ , we have  $\tilde{a}_1(\theta) = \alpha_1 q(\theta) + \beta_1$  and  $\tilde{a}_2(\theta) = \alpha_2 q(\theta) + \beta_2$  with  $\alpha_i$  and  $\beta_i$  real numbers,  $i = 1, 2$ . As we can take without loss of generality that  $q(0) = 0$ , and since  $\tilde{a}_1(0) = \tilde{a}_2(0) = 0$ , we get that  $\beta_1 = \beta_2 = 0$ . Hence in this case equation (1.1) takes the form

$$\frac{d\rho}{d\theta} = q'(\theta)(\alpha_1 \rho^2 + \alpha_2 \rho^3),$$

which is an equation of separable variables.

Assume now that  $d_1$  and  $d_2$  are coprime. Again by Lemma 2.2(c), we have that  $\deg q = 1$  (or it would be a common divisor of  $d_1$  and  $d_2$ ). Thus, the differential equation (1.1) has a center which is  $\alpha$ -symmetric, see [14].  $\square$

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